Option Valuation: The binomial model

Road Map/key Ideas

- Determinants of option price
- How to price options using a binomial model

Option pricing

Some usage of options

- Trade on views of directions of the underlying asset or volatility
- Hedging risks
- Price assets that have embedded options such as the callable bonds. Callable bonds can be priced as a vanilla bond - a call option.
- Price "real option". Future investment opportunity of the firm can be viewed as the firm having an option to invest in the future

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Determinants of Option Prices

	call	put
• Stock price		
• Strike price		
• Volatility		
• Time to maturity		
• Interest rate		
• Dividends		

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Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price		
• Volatility		
• Time to maturity		
• Interest rate		
• Dividends		

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Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility		
• Time to maturity		
• Interest rate		
• Dividends		

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Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility	+	+
• Time to maturity		
• Interest rate		
• Dividends		

Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility	+	+
• Time to maturity	+	+
• Interest rate		
• Dividends		

Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility	+	+
• Time to maturity	+	+
• Interest rate	+	-
• Dividends	-	+

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Bounds on option price

- Consider 2 portfolios:1. Long a call option, and 2. long the stock and borrow PV(X+D). D is the dividend of the stock.

- At expiration of the option each portfolio pay off is

1. Call = Max (S_T-X,0)

2. S_T + D – (X+D) = S_T –X

- Therefore, C_t >= S_t –PV(X+D)

- And S_t >=C_t >= S_t –PV(X+D)

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Bounds on option price

- Consider 2 portfolios:1. Long a put option, and 2. short sell the stock and lend PV(X+D). D is the dividend of the stock.
- At expiration of the option each portfolio pay off is

1. Put = Max $(X-S_T,0)$
2. $X+D-(S_T + D) = X-S_T$

- Therefore, $P_t >= PV(X+D)-S_t$
- And $X >=P_t >= Max(PV(X+D)-S_t, 0)$

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Option pricing: binomial tree

- An option pricing model assumes a distribution of how the underlying asset (stock) evolves in the future.
- The binomial model assumes the binomial distribution.
- Key innovation: The payoff of an option can be replicated dynamically by a portfolio that consists of some position in the underlying stock and borrowing/lending.

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Option pricing: binomial tree

Simple example: A call option expiring in 1 year with a strike price or $150. The risk-free rate is 10%

0.5 $200

Suppose S_0 = $100 & S_1 =

0.5 $ 50

What is the pay off of the option at maturity?

		S_1	C_1
	0.5	200	50
S_0			
	0.5	50	0

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Option pricing: binomial tree

What if we form a portfolio that invests in Δ shares the underlying stock and borrow some money?

The portfolio is $\Delta S_0 - D = C_0$ where D is a money market account (borrowing/lending)

pay off

1/3 * 200 - $15.15*1.1 = 66.67-16.17 =50

$1/3S_0$-$15.15

1/3 * 50 - $15.15*1.1 = 16.67-16.17 = 0

This is the same as pay off from the call option!

So, $1/3S_0$-$15.15 = C_0

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Option pricing: binomial tree

Form a portfolio of Δ shares of stock and short one call option.

ΔS_0-C

$\Delta(200)$ - \$50

$\Delta(50)$ - \$0

Select the "hedge ratio" , Δ, so that the payoff of the portfolio is the same in either state of the world:

Find Δ:	$\Delta(200) - 50$	=	$\Delta(50)$
	$\Delta(150)$	=	50
	Δ	=	(1/3)

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Option pricing: binomial tree

	1/3(200) - 50	=	\$16.67
Payoff	1/3(50)	=	\$16.67

The hedged portfolio is riskless, thus it should get a return equal to the riskless rate.The Value of this portfolio at time 0=

$$= \frac{\$16.67}{1.10} = \$15.15$$

The value of the hedged portfolio is

$$(1/3)S_0 - C_0 = \$15.15$$

$$\text{and } S_0 = \$100 \quad \rightarrow \quad C_0 = \$18.18$$

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Option pricing: binomial tree

This means that we can replicate the pay off of this call option investing in a portfolio that consists of 1/3 of the the stock and borrowing $15.15.

pay off

$1/3 * 200 - \$15.15*1.1 = 66.67-16.17 = 50$

$1/3S_0-\$15.15$

$1/3 * 50 - \$15.15*1.1 = 16.67-16.17 = 0$

This is the same as pay off from the call option!

So, $1/3S_0-\$15.15 = C_0 = 18.18$

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Option pricing (Algebra)

<u>More generally</u>

uS

q

S

dS

C_u = max [0, uS-X]

q

C

C_d = max[0, dS-X]

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Option pricing (Algebra)

Stocks in the binomial model have equal probability of going up or down. Form a portfolio of Δ shares of stock and one call option (short).

$\Delta uS - C_u$

$\Delta S - C$

$\Delta dS - C_d$

Pick Δ: so that the portfolio is riskless i.e.,

$\Delta uS - C_u \quad = \quad \Delta dS - C_d$

$\Delta(u-d)S \quad = \quad C_u - C_d \quad \longrightarrow \quad \Delta = \dfrac{C_u - C_d}{(u-d)S} \left(\approx \dfrac{dC}{dS}\right)$

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Option pricing (Algebra)

Value of Portfolio at t = 0 : $\Delta S - C$

Value of Portfolio at t = 1 : $\Delta\, uS - C_u$ (for sure)

$$\longrightarrow \quad \frac{\Delta uS - C_u}{1 + r_f} = \Delta S - C$$

- Substituting Δ and defining $\quad p = \dfrac{r^* - d}{u - d}; \quad r^* = 1 + r_f$
- We have $\quad C = \dfrac{pC_u + (1-p)C_d}{1 + r_f}$

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Option Valuation: Binomial probability model

For

$$S_0 \begin{cases} S_u = uS \\ S_d = dS \end{cases} \qquad C \begin{cases} C_u \\ C_d \end{cases}$$

r_f = risk free rate for period t

$$C = \frac{pC_u + (1-p)C_d}{1+r_f}, \text{ where } \quad p = \frac{r^* - d}{u-d}; \quad r^* = 1 + r_f$$

If S_u and S_d are given then $d = \frac{S_d}{S_o} \qquad u = \frac{S_u}{S_o}$

If the volatility is given then $d = e^{-\sigma\sqrt{t}}\ ; u = e^{\sigma\sqrt{t}}$

For put options, substitute C with P

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Observations

- The option value does not depend on the rate of return of the underlying asset
- *p* can be interpreted as the probability that the option is going to end up in-the-money

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Option Valuation: Binomial Method

- Initial Example: S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%

$$p = \frac{1.1 - 0.5}{2.0 - 0.5} = \frac{0.6}{1.5} = 0.4$$

$$\longrightarrow C = \frac{0.4(\$50) + 0.6(\$0)}{1.10} = \$18.18$$

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Concept check

- S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%. What is the price of a call option with X=100 ?

G) 18.18

Y) 30

R) 36.36

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Concept check

- S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%. What is the price of a put option with X=100 ?

G) 18.18

Y) 30

R) 27.27

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Concept check: solution

- S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%. What is the price of a put with X=100

$$p = \frac{1.1 - 0.5}{2.0 - 0.5} = \frac{0.6}{1.5} = 0.4$$

$$P_u = \max(0{,}100 - 200) = 0$$

$$P_d = \max(0{,}100 - 50) = 50$$

$$P = \frac{0.4(\$0) + 0.6(\$50)}{1.10} = \$27.27$$

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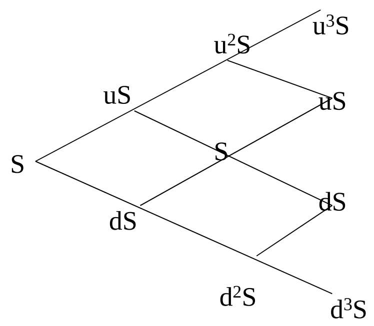
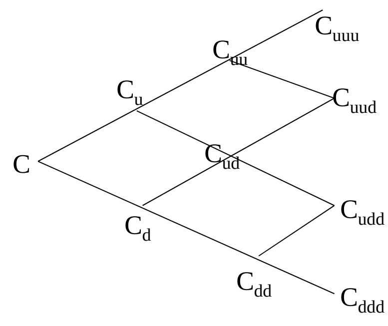
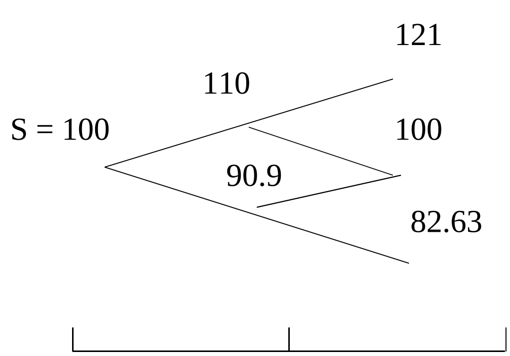
Binomial Method: Multi-step binomial model





Example: Call Option X = $100



u = 1.10

d = 0.909

r* = 1.02

1 year

Sample: Call Option   X = $100

Price an at-the-money call option with X = $100

$$p = \frac{r^* - d}{u - d} = \frac{1.02 - 0.909}{1.10 - 0.909} = 0.5812$$

C_u → 21, 0

$$\rightarrow C_u = \frac{p(21) + (1-p)\,0}{1.02}$$

= $11.97

C_d → 0, 0

$$\rightarrow C_d = \frac{p(0) + (1-p)\,0}{1.02} \quad = \quad \$0$$

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Sample: Call Option   X = $100

C_0 → C_u = (11.97), C_d = (0)

$$C_0 = \frac{pC_u + (1-p)C_d}{1.02} = \frac{0.5812(11.97) + (1-0.5812)(0)}{1.02} = 6.82$$

Option is worth $6.82

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Summary

- Determinants of option price
- How to price options using a binomial model