

Solution: Exercise 1 (Part 2)

1. Determine whether the statement forms are logically equivalent. In each case, construct a truth table to justify your answer.

(a) $(p \rightarrow q) \rightarrow (q \rightarrow p)$ and $(p \vee q) \rightarrow (p \wedge q)$

(b) $\sim p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$

(c) $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$

(d) $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$

Answer:

(a) $(p \rightarrow q) \rightarrow (q \rightarrow p)$ and $(p \vee q) \rightarrow (p \wedge q)$

Truth table:

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \vee q$	$p \wedge q$	$(p \rightarrow q) \rightarrow (q \rightarrow p)$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	F	T
F	F	T	T	F	F	T	F

Since rows 3 and 4 of the truth table for $(p \rightarrow q) \rightarrow (q \rightarrow p)$ and $(p \vee q) \rightarrow (p \wedge q)$ have different truth values, then these statement forms are not logically equivalent. ■

(b) $\sim p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$

p	q	r	$\sim p$	$q \rightarrow r$	$p \vee r$	$\sim p \rightarrow (q \rightarrow r)$	$q \rightarrow (p \vee r)$
T	T	T	F	T	T	T	T
T	T	F	F	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	T

Since all rows 8 (all possible cases for truth values of p, q, r) in the truth table for $\sim p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ have the same truth values, then these statement forms are logically equivalent. ■

(c) $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$

Truth table:

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$	$p \leftrightarrow q$
T	T	T	T	F	F	T
T	F	T	F	T	T	F
F	T	T	F	T	T	F
F	F	F	F	T	F	T

Since all 4 rows of the truth table for $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$ have different truth values, then these statement forms are not logically equivalent. ■

(d) Truth table for $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \leftrightarrow r$	$(p \leftrightarrow q) \leftrightarrow r$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	F	T	F	F
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	T	F	F

Since all rows 8 (all possible cases for truth values of p, q, r) in the truth table for $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$ have the same truth values, then these statement forms are logically equivalent. ■

2. Determine whether or not the statement $p \wedge q \rightarrow (p \rightarrow q)$ is a tautology or a contradiction.

Answer:

Truth table for $p \wedge q \rightarrow (p \rightarrow q)$

p	q	$p \wedge q$	$p \rightarrow q$	$p \wedge q \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Since all the rows (all possible cases for the truth values for p and q) in the truth table for $p \wedge q \rightarrow (p \rightarrow q)$ have the **true** truth value, then this statement form is a **tautology**. ■

3. Let p, q and r be statements such that $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ is **true**. Determine the truth value of $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$.

Answer: False

Truth table for $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \wedge (q \leftrightarrow r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	F	F
F	F	F	T	T	T

Notice from the above truth table that there are only 2 cases for p, q, r that make $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ true. These 2 cases are in the first and the last rows:

- (i) p, q, r are all true or
- (ii) p, q, r are all false.

That is, we only need to consider these two cases for $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$, which can be shown to be **false** for both cases (see the table below).

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \vee q \vee r$	$\sim p \vee \sim q \vee \sim r$	$(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$
T	T	T	F	F	F	T	F	F
F	F	F	T	T	T	F	T	F

Hence, when $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ is **true**, the truth value of $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$ is always **false**. ■

4. Consider the following statement.

If I eat spicy food and I drink beer, then I feel sick or I have a bad dream.

- (a) Write the **negation** of the above statement.
 (b) Write the **contrapositive**, **inverse**, and **converse** of the above statement.

Answer:

Let p be “I eat spicy food ,”

q be “I drink beer,”

r be “I feel sick,”

s be “I have a bad dream.”

Then the statement can be written as

$$(p \wedge q) \rightarrow (r \vee s).$$

- (a) The **negation** :

$$\sim [(p \wedge q) \rightarrow (r \vee s)] \equiv (p \wedge q) \wedge \sim (r \vee s) \equiv (p \wedge q) \wedge (\sim r \wedge \sim s)$$

“*I eat spicy food and I drink beer, but (and) I do not feel sick and I do not have a bad dream.*”

- (b) **Contrapositive:**

$$\sim (r \vee s) \rightarrow \sim (p \wedge q) \equiv (\sim r \wedge \sim s) \rightarrow (\sim p \vee \sim q)$$

“*If I do not feel sick **and** I do not have a bad dream, then I do not eat spicy food **or** I do not drink beer.*”

Inverse:

$$\sim (p \wedge q) \rightarrow \sim (r \vee s) \equiv (\sim p \vee \sim q) \rightarrow (\sim r \wedge \sim s)$$

“*If I do not eat spicy food **or** I do not drink beer, then I do not feel sick **and** I do not have a bad dream.*”

Converse

$$(r \vee s) \rightarrow (p \wedge q)$$

“*If I feel sick or I have a bad dream, then I eat spicy food and I drink beer.*”



5. Use truth tables to determine whether the argument forms are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

$$\begin{array}{l} (a) \quad p \rightarrow \sim q \\ \quad \sim q \rightarrow p \\ \therefore p \vee \sim q \end{array}$$

$$\begin{array}{l} (b) \quad r \vee \sim q \\ \quad p \rightarrow q \\ \quad \sim r \\ \therefore \sim p \end{array}$$

- (c) If I answer the quiz correctly, then I understand the class material.
I understand the class material.
 $\therefore$ I answer the quiz correctly.

Answer:

$$\begin{array}{l} (a) \quad p \rightarrow \sim q \\ \quad \sim q \rightarrow p \\ \therefore p \vee \sim q \end{array}$$

Truth table:

			Premises		Conclusion
p	q	$\sim q$	$p \rightarrow \sim q$	$\sim q \rightarrow p$	$p \vee \sim q$
T	T	F	F	T	
T	F	T	T	T	T
F	T	F	T	T	F
F	F	T	T	F	

←-- critical row

←-- critical row: This has true premises but false conclusion.

Notice that the columns 4 and 5 consist of the truth values of premises and rows 2 and 3 are critical rows (premises are all true). Since the row 3 (which is a critical row) has true premises with false conclusion, then this argument is **invalid**. ■

$$\begin{array}{l} (b) \quad r \vee \sim q \\ \quad p \rightarrow q \\ \quad \sim r \\ \therefore \sim p \end{array}$$

Truth table:

Premises						Conclusion	
p	q	r	$\sim q$	$r \vee \sim q$	$p \rightarrow q$	$\sim r$	$\sim p$
T	T	T	F	T	T	F	F
T	T	F	F	F	T	T	F
T	F	T	T	T	F	F	F
T	F	F	T	T	F	T	F
F	T	T	F	T	T	F	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	T	T

←

←-- critical row

Notice that the columns 5, 6 and 7 give the truth values of premises and only row 8 (the last row) is the critical row. Since this critical row has all true premises with true conclusion, then this argument is **valid**. ■

(c) Let p be "I answer the quiz correctly"; and q be "I understand the class material."

If I answer the quiz correctly, then I understand the class material. $p \rightarrow q$

I understand the class material. $\equiv q$

$\therefore$ I answer the quiz correctly. $\therefore p$

Truth table:

Premises				Conclusion	
p	q	$p \rightarrow q$	q	p	
T	T	T	T	T	←-- critical row
T	F	F	F		
F	T	T	T	F	←-- critical row: This has true premises but false conclusion.
F	F	T	F		

Notice that the columns 3 and 4 consist of the truth values of premises and rows 1,3 are the critical rows. Since there is a critical row (row 3) that has true premises with false conclusion, then this argument is **invalid**. ■

6. Consider the following premises.

- (i) It is not sunny this afternoon and it is colder than yesterday.
- (ii) We will go swimming only if it is sunny.
- (iii) If we do not go swimming, then we will take a canoe trip.
- (iv) If we take a canoe trip, then we will be home by sunset.

From the above premises (i)-(iv), does the conclusion that *we go home by sunset* make a valid argument? Explain your answer by using rules of inferences.

Answer:

Let p be the statement "It is sunny this afternoon,"

q the statement "It is colder than yesterday,"

r the statement "We will go swimming,"

s the statement "We will take a canoe trip," and t the statement "We will be home by sunset."

Then the argument consists of the following premises and conclusion.

$$\sim p \wedge q$$

$$r \rightarrow p$$

$$\sim r \rightarrow s$$

$$s \rightarrow t$$

$$\therefore t$$

We construct an argument to show that our premises lead to the desired conclusion as follows.

(1) Simplification : premise (i)

$$\sim p \wedge q$$

$$\therefore \sim p$$

(2) Modus tollens: using premise (ii) and (1)

$r \rightarrow p$ premise (ii)

$\sim p$ from (1)

$\therefore \sim r$

(3) Modus ponens: using premise (iii) and (2)

$\sim r \rightarrow s$ premise (iii)

$\sim r$ from (2)

$\therefore s$

(4) Modus ponens: using premise (iv) and (3)

$s \rightarrow t$ premise (iv)

s from (3)

$\therefore t$

Therefore, we can conclude that “we will be home by sunset,” from rules of inference used in (1)-(4).

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p , q , r , s , and t , such a truth table would have 32 rows. ■