

Solving Inequalities: II

TU152: Fundamental Mathematics

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Absolute Value

Definition (Absolute Value)

For any real number x , the absolute value of x , denoted $|x|$, is defined as follows:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

From the definition, the absolute value $|x|$ can be viewed as “the distance of x from zero” on the number line. E.g.

- “Equality”: Both -3 and 3 are three units from zero. This implies $|-3| = |3| = 3$.
- “Less than”: The solution to $|x| < 3$ consists of the points that are less than three units away from zero, i.e.

$$-3 < x < 3.$$

The solution set to $|x| \leq 3$ includes the solutions to $|x| < 3$ and $x = -3, 3$.

- “Greater than”: The solution to $|x| > 3$ consists of the points that satisfy either one of the following two inequalities

$$x < -3 \quad \text{or} \quad x > 3.$$

The solution set to $|x| \geq 3$ includes the solutions to $|x| > 3$ and $x = -3, 3$.

Properties of Absolute-Value Inequalities

Theorem 1: Properties of Absolute-Value Inequalities

Let x be a real number and a be a positive real number.

(1) $|x| < a$ if and only if $-a < x < a$.

(2) $|x| \leq a$ if and only if $-a \leq x \leq a$.

(3) $|x| > a$ if and only if $x < -a$ or $x > a$.

(4) $|x| \geq a$ if and only if $x \leq -a$ or $x \geq a$.

Example Find the solution set for each of the inequalities.

• $|x + 1| \leq -5$

• $|2x - 3| > -5$

Example Find the solution set for each of the inequalities.

- $|x + 1| \leq 5$

- $|2x - 3| > 5$

Properties of Absolute-Value Inequalities

Theorem 2: Properties of Absolute-Value Inequalities

Let x and y be real numbers.

- (1) If $x \neq 0$, then $|x| > 0$.
- (2) $|x| = 0$ if and only $x = 0$
- (3) $|-x| = |x|$, $|x| \geq |x|$, $|x| \leq |x|$
- (4) $-|x| \leq x \leq |x|$
- (5) $|x + y| \leq |x| + |y|$
- (6) $|x - y| \leq |x| + |y|$
- (7) $||x| - |y|| \leq |x - y|$
- (8) $|xy| = |x||y|$
- (9) $\left| \frac{x}{y} \right| \leq \frac{|x|}{|y|}$
- (10) $|x| < |y|$ if and only if $x^2 < y^2$, $|x| \leq |y|$ if and only if $x^2 \leq y^2$
- (11) $|x|^2 = x^2$

Example:

Let x and y be real numbers. Prove that if $|x| < 3$ and $y > 0$, $|x + y| < y + 3$.

Example: Let x be any real number with $|x| < 3$. Prove that $|x^2 - 1| < 4|x + 1|$.

Example: Let x and y be positive real numbers. Show that if $|x + 1| < x + y$, then

$$1 - y < \frac{y^2 - 1}{2x}.$$

Solving Absolute-Value Inequalities

Example: Find the solution set for the inequality:

$$2 < |2x - 1| \leq 5.$$

Example: Find the solution set for the inequality:

$$|x + 2| \leq |2x + 3|.$$

Example: Find the solution set for the inequality:

$$|x + 5| \geq 3 + |x|.$$

Example: Find the solution set for following inequality.

$$|x^2 + x - 6| < 6$$

Example: Find the solution set for following inequality.

$$\frac{x^2}{|x| - 1} \leq |x + 1|$$