Solution: Assignment 6

1. Find all the asymptotes of the following relation:

$$r = \left\{ (x, y) \in X \times Y \,\middle|\, \quad yx^2 = (1-x)^3 \right\}.$$

<u>**Answer:**</u> Notice that

$$(1-x)^3 = (1-x)(1-2x+x^2) = -x^3 + 3x^2 - 3x + 1$$

and we can write $yx^2 = (1-x)^3$ as $y = f(x)$ where

$$f(x) = \frac{(1-x)^3}{x^2} = \frac{-x^3 + 3x^2 - 3x + 1}{x^2}$$

for $x \neq 0$.

<u>**(i) Horizontal Asymptote**</u>
Since

$$\lim_{x \to \infty} f(x) = \frac{(1-x)^3}{x^2} = -\infty$$

and

$$\lim_{x \to -\infty} f(x) = \frac{(1-x)^3}{x^2} = \infty,$$

which are not constant, then there is no horizontal asymptote. We have used the fact that the limit of rational function with higher degree in the numerator will have limit approaching $\pm\infty$.

<u>**(ii) Vertical Asymptote**</u>
Notice that the only value that make $f(x)$ undefined is $x = 0$. From

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{-x^3 + 3x^2 - 3x + 1}{x^2} = \infty$$

where we have used the fact that the limit is in the form $\frac{-0+0-0+1}{0} = \frac{1}{0}$ which approaches ∞. Hence, the vertical asymptote is $x = 0$.

<u>**(iii) Slant Asymptote**</u>
Notice that degree of numerator is greater than the degree of the denominator by one, there is a slant asymptote which can be found as follows. Using the long division we have

$$\begin{array}{r|l} & -x+3 \\ x^2 & -x^3+3x^2-3x+1 \\ & \underline{-x^3} \\ & 3x^2-3x+1 \\ & \underline{3x^2} \\ & -3x+1 \end{array}$$

we have

$$y = \frac{(1-x)^3}{x^2} = \frac{-x^3 + 3x^2 - 3x + 1}{x^2} = (-x+3) + \frac{-3x+1}{x^2}.$$

Note that the long division stops as soon as the degree of the remainder is smaller than the divisor. Note also that this can be obtained directly by distributing the term $\frac{1}{x^2}$. Therefore the numerical the slant asymptote is the line

$$y = -x + 3$$

which can also be checked from the definition:

$$\lim_{x\to\infty}[f(x) - (-x+3)] = \lim_{x\to\infty}\left[\left((-x+3) + \frac{-3x+1}{x^2}\right) - (-x+3)\right] = \lim_{x\to\infty}\left[\frac{-3x+1}{x^2}\right] = 0.$$

From(i)-(iii), the aysmptotes for the graph $y = f(x)$ are:
the vertical asymptote is $x = 0$ and
the slant asymptote $y = -x + 3$. ■

2. Consider the relation f defined from X to Y, $X, Y \subseteq \mathbb{R}$,

$$f = \left\{(x, y) \in X \times Y \middle| \quad yx^3 = x^2 - 1\right\}.$$

   (a) Find the domain of f.
   (b) Find x-intercepts and y-intercepts (if any).
   (c) Determine the symmetry of f.
   (d) Find the horizontal and vertical asymptotes for f (if any).
   (e) Find the critical number of f. Determine the intervals on which f is increasing and decreasing. Determine the relative extrema (maximum and minimum) of f.
   (f) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).
   (g) Sketch the curve of f.

Answer:

   (a) Notice that we can rearrange as $y = \frac{x^2-1}{x^3}$. So, the domain of f (which consists of possible values of x) is $\mathbb{R} - \{0\}$.
   (b) x-intercepts: setting $y = 0$ gives $0 = \frac{x^2-1}{x^3}$, which occurs if and only if

$$x^2 - 1 = 0 \quad \Leftrightarrow \quad (x-1)(x+1) = \quad \Leftrightarrow \quad x = -1, 1.$$

   So x-intercepts are $(-1, 0), (1, 0)$.
   To find y-intercepts, set $x = 0$. However, 0 is not in the domain. So There is **no y-intercepts**.
   (c) Determine the symmetry of f.
   Notice that f is a function and can be written as $f(x) = \frac{x^2-1}{x^3}$. Hence, it is not symmetric about the x-axis. We will look at the symmetry by using $f(-x)$.

$$f(-x) = \frac{(-x)^2 - 1}{(-x)^3} = -\frac{x^2 - 1}{x^3}.$$

   Since $f(-x) = -f(x)$, f is an odd function and it is symmetric about the origin.
   Note that $f(-x) \neq f(x)$, it is not symmetric about the y-axis.

(d) Find the horizontal and vertical asymptotes for f (if any).
Horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-1}{x^3} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-1}{x^3} = 0$$

So, the horizontal asymptote is $y = 0$.
Vertical asymptotes:

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{x^2-1}{x^3} = -\infty$$

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{x^2-1}{x^3} = +\infty.$$

So, the vertical asymptote is $x = 0$.

(e) Find the critical number of f.

$$f'(x) = \frac{x^3(2x) - 3x^2(x^2-1)}{x^6} = \frac{2x^4 - 3x^4 + 3x^2}{x^6} = \frac{-x^2+3}{x^4}$$

The critical number x occurs when $f'(x) = 0$ or $f'(x)$ does not exist.

$$f'(x) = 0 \quad \Leftrightarrow \quad -x^2+3 = 0 \quad \Leftrightarrow \quad x = -\sqrt{3}, \sqrt{3}.$$

Note that $f'(x)$ does not exist when $x = 0$, which is not in the domain, so 0 is not a critical number. Hence, the critical numbers are $x = -\sqrt{3}, \sqrt{3}$.

(f) Determine the intervals on which f is increasing and decreasing. Determine the relative extrema (maximum and minimum) of f.
From

$$f'(x) = \frac{-x^2+3}{x^4} = -\frac{(x+\sqrt{3})(x-\sqrt{3})}{x^4}$$

- Decreasing: $f'(x) < 0$ when $-(x+\sqrt{3})(x-\sqrt{3}) < 0$ or $(x+\sqrt{3})(x-\sqrt{3}) > 0$
- Increasing: $f'(x) > 0$ when $-(x+\sqrt{3})(x-\sqrt{3}) > 0$ or $(x+\sqrt{3})(x-\sqrt{3}) < 0$

	$x < -\sqrt{3}$	$-\sqrt{3} < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of $f'(x) = \frac{-(x+\sqrt{3})(x-\sqrt{3})}{x^4}$	$-$	$+$	$-$
$-(x+\sqrt{3})(x-\sqrt{3})$	$-(-)(-)$	$-(+)(-)$	$-(+)(+)$

I.e., $f'(x) < 0$ when $x < -\sqrt{3}$ and $x > \sqrt{3}$. Also, $f'(x) < 0$ for $x \in (-\sqrt{3}, \sqrt{3})$ or $x \in (-\sqrt{3}, 0) \cup (0, \sqrt{3})$(note: $x = 0$ is not in the domain)

- Decreasing interval: $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$
- Increasing interval: $(-\sqrt{3}, 0) \cup (0, \sqrt{3})$.

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = \sqrt{3}$ ("+" changes to "−" at $x = \sqrt{3}$) and $f(\sqrt{3}) = \frac{2}{3\sqrt{3}}$ is the relative maximum
- relative minimum occurs at $x = -\sqrt{3}$ ("−" changes to "+" at $x = -\sqrt{3}$) and $f(-\sqrt{3}) = -\frac{2}{3\sqrt{3}}$ is the relative minimum.

(g) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).
From $f'(x) = \frac{-x^2+3}{x^4}$,

$$f''(x) = \frac{x^4(-2x) - (3-x^2)(4x^3)}{x^8} = \frac{-2x^5 - 12x^3 + 4x^5}{x^8} = \frac{2(x^5 - 6x^3)}{x^8} = \frac{2(x^2-6)}{x^5}.$$

So $f''(x) = 0$ when $\frac{x^2-6}{x^5} = \frac{(x-\sqrt{6})(x+\sqrt{6})}{x^5} = 0$ or $x = \pm\sqrt{6}$ and $f''(x)$ is undefined when $x = 0$. To solve the inequalities , we will also consider the denominator (because x^5 changes sign for $x < 0$ and $x > 0$).

Interval of x	$(-\infty, -\sqrt{6})$	$(-\sqrt{6}, 0)$	$(0, \sqrt{6})$	$(\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2(x^2-6)}{x^5}$ $\frac{(x-\sqrt{6})(x+\sqrt{6})}{x^5}$	$-$ $\frac{(-)(-)}{(-)}$	$+$ $\frac{(-)(+)}{(-)}$	$-$ $\frac{(-)(+)}{(+)}$	$+$ $\frac{(+)(+)}{(+)}$

- Concave up: $f'' > 0$ when $x \in (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$.
- Concave down: $f'' < 0$ when $x \in (-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$.

The points of inflection are $(\sqrt{6}, f(\sqrt{6})) = (\sqrt{6}, \frac{5}{6\sqrt{6}})$ and $(-\sqrt{6}, f(-\sqrt{6})) = (-\sqrt{6}, -\frac{5}{6\sqrt{6}})$ because the curve changes the concavity at these points (note that 0 is not an inflection point because it is not in the domain).

(h) Sketch the curve of f.

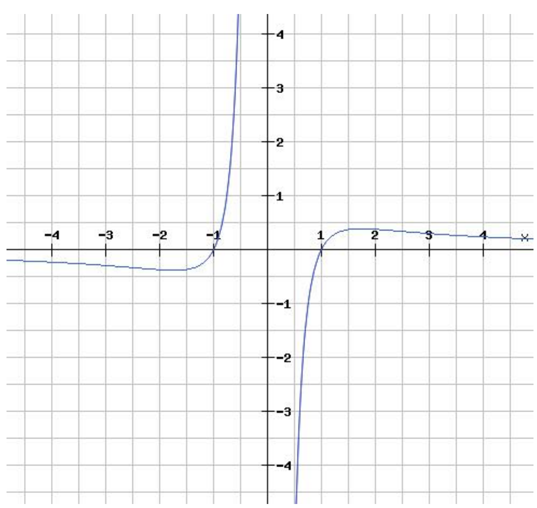

Figure 1: $f = \{(x, y) | yx^3 = x^2 - 1\}$

Optional Problems

1. Find the critical numbers of the given functions on $(-\infty, \infty)$.

(a) $f(x) = x^3 - 3x^2 + 3x - 1$
(b) $f(x) = \frac{x^2}{x^2+2}$
(c) $f(x) = e^{-x} + 2x$
(d) $f(x) = -x + \sin(x)$
(e) $f(x) = x^2 - 8\ln(x)$

(a) $f(x) = x^3 - 3x^2 + 3x - 1$
<u>Answer:</u> $f'(x) = 3x^2 - 6x + 3$

$$f'(x) = 0 \quad \Leftrightarrow \quad 3(x^2 - 2x + 1) = 3(x-1)^2 = 0 \quad \Leftrightarrow \quad (x-1) = 0 \quad \Leftrightarrow \quad x = 1$$

Therefore, the only critical number of $f(x)$ is $x = 1$.

(b) $f(x) = \frac{x^2}{x^2+2}$
<u>Answer:</u>

$$f'(x) = \frac{(x^2+2)(2x) - x^2(2x)}{(x^2+2)^2} = \frac{2x^3 + 4x - x^3}{(x^2+2)^2} = \frac{x^3 + 4x}{(x^2+2)^2}$$

$f'(x) = 0$ when $x^3 + 4x = 0 \Longrightarrow x(x^2+4) \Longrightarrow x = 0$.
Therefore, the only critical number of $f(x)$ is $x = 0$.

(c) $f(x) = e^{-x} + 2x$
<u>Answer:</u>

$$f'(x) = e^{-x} + 2x = -e^{-x} + 2$$

$f'(x) = 0$ when $-e^{-x} + 2 = 0 \Longrightarrow e^{-x} = 2 \Longrightarrow e^x = 1/2 \Longrightarrow x = \ln(1/2)$ or $x = -\ln(2)$.
Therefore, the only critical number of $f(x)$ is $x = -\ln(2)$.

(d) $f(x) = -x + \sin(x)$
<u>Answer:</u>

$$f'(x) = -1 + \cos(x)$$

$f'(x) = 0$ when $-1 + \cos(x) = 0 \Longrightarrow \cos(x) = 1 \Longrightarrow x = 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$.
Therefore, the critical numbers of $f(x)$ are $x = 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$.

(e) $f(x) = x^2 - 8\ln(x)$
<u>Answer:</u>

$$f'(x) = 2x - \frac{8}{x}$$

$f'(x) = 0$ when $2x - \frac{8}{x} = 0 \Longrightarrow x = \frac{4}{x} \Longrightarrow x^2 = 4 \Longrightarrow x = \pm 2$.
Therefore, the critical numbers of $f(x)$ are $x = -2$ and $x = 2$.

2. Determine the intervals on which the given function f is increasing and the interval on which f is decreasing.

(a) $f(x) = x^2 + 6x - 1$ (b) $f(x) = x^4 - 4x^3 + 9$ (c) $f(x) = x^2 e^{-x}$

<u>Answer:</u> The fuction $f(x)$ is increasing when $f'(x) > 0$ and $f(x)$ is decreasing when $f'(x) < 0$.

(a) $f(x) = x^2 + 6x - 1 \Longrightarrow f'(x) = 2x + 6$

$$f'(x) = 2x + 6 > 0 \quad \Longrightarrow \quad x > -3$$

so $f(x)$ is increasing on the interval $[-3, \infty)$, and

$$f'(x) = 2x + 6 < 0 \quad \Longrightarrow \quad x < -3$$

so $f(x)$ is decreasing on the interval $(\infty, -3]$.

(b) $f(x) = x^4 - 4x^3 + 9 \Longrightarrow f'(x) = 4x^3 - 12x^2$

$$f'(x) = 4x^3 - 12x^2 > 0 \quad \Longrightarrow 4x^2(x-3) > 0 \quad \Longrightarrow x > 3$$

so $f(x)$ is increasing on the interval $[3, \infty)$, and

$$f'(x) = 4x^3 - 12x^2 < 0 \quad \Longrightarrow 4x^2(x-3) < 0 \quad \Longrightarrow x < 3$$

so $f(x)$ is decreasing on the interval $(\infty, 3]$.

(c) $f(x) = x^2e^{-x} \Longrightarrow f'(x) = -x^2e^{-x} + 2xe^{-x} \Longrightarrow f'(x) = xe^{-x}(2-x)$,
since $e^{-x} > 0$, $f'(x) = 0 \Longrightarrow x = 0$ or $x = 2$:

	$x < 0$	$0 < x < 2$	$2 < x$
Sign of $f'(x) = xe^{-x}(2-x)$	$-$	$+$	$-$
$(x)(e^{-x})(2-x)$	$(-)(+)(+)$	$(+)(+)(+)$	$(+)(+)(-)$

so $f(x)$ is increasing $(f'(x) = +)$ on the interval $[0, 2]$,
and $f(x)$ is decreasing $(f'(x) = -)$ on the interval $(\infty, 0] \cup [2, \infty)$.

3. Use the *First Derivative Test* to find the relative extrema of the given function.
(a) $f(x) = x^3 - 3x$ (b) $f(x) = x^3 + x - 3$ (c) $\frac{x^2+3}{x+1}$ (d) $f(x) = x^2 - 2|x|$

Answer:

(a) $f(x) = x^3 - 3x \Longrightarrow f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$,
$f'(x) = 0 \Longrightarrow x = \pm 1$ are critical numbers:

	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of $f'(x) = 3(x-1)(x+1)$	$+$	$-$	$+$
$(x-1)(x+1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = -1$ ("$+$" changes to "$-$" at $x = -1$)
and $f(-1) = (-1)^3 - 3(-1) = 2$ is the relative maximum
- relative minimum occurs at $x = 1$ ("$-$" changes to "$+$" at $x = 1$) and
$f(1) = 1^3 - 3(1) = -2$ is the relative minimum.

(b) $f(x) = x^3 + x - 3 \Longrightarrow f'(x) = 3x^2 + 1 \neq 0, \forall x \in (-\infty, \infty)$.

$f'x)$ is well-defined for any x and $f'(x) \neq 0$ imply that there is *no critical number* for $f(x)$. Since a relative extremum can only occur at a critical number and there is no critical number, therefore, we conclude that **there is no relative extrema** for this function.

(c) $f(x) = \frac{x^2+3}{x+1}$ Note that the domain of this function is all real number except for $x = -1$: $D = \mathbb{R}/\{-1\}$
$\Longrightarrow f'(x) = \frac{(x+1)(2x)-(x^2+3)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$,
$f'(x) = 0 \Longrightarrow (x+3)(x-1) = 0 \quad \Longrightarrow x = -3, 1$ are the critical numbers.

	$x < -3$	$-3 < x < 1 (x \neq -1)$	$1 < x$
Sign of $f'(x)$	$+$	$-$	$+$
$(x+3)(x-1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = -3$ ("+" changes to "−" at $x = -3$)
and $f(-3) = \frac{(-3)^2+3}{-3+1} = -6$ is the relative maximum
- relative minimum occurs at $x = 1$ ("−" changes to "+" at $x = 1$) and
$f(1) = \frac{1^2+3}{1+1} = 2$ is the relative minimum.

Notice that it is possible to have a " relative" minimum *larger* than a " relative" maximum.

(d) Note that we can write $f(x) = x^2 - 2|x|$ as:

– For $x < 0$, $f(x) = x^2 + 2x$
– For $x \geq 0$, $f(x) = x^2 - 2x$.

(i) The critical numbers:

– For $x < 0$, $f'(x) = 2x + 2$
– For $x > 0$, $f'(x) = 2x - 2$,

and $f'(x) = 0$ when $2x + 2 \;\Rightarrow\; x = -1$ or $2x - 2 = 0 \;\Rightarrow\; x = 1$. Note that $f'(x)$ does not exist when $x = 0$. Hence, the critical numbers are $x = -1, 1, 0$.

(ii) Relative extrema: Relative extrema can occur only at the critical points: $x = -1, 1, 0$. So we consider the subintervals:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of $f'(x)$	$f'(x) = 2x + 2$ $-$	$f'(x) = 2x + 2$ $+$	$f'(x) = 2x - 2$ $-$	$f'(x) = 2x - 2$ $+$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = 0$ ("+" changes to "−" at $x = 0$)
and $f(0) = 0$ is the relative maximum
- relative minima occurs at $x = -1, 1$ ("−" changes to "+" at $x = -1$, 1) and
$f(-1) = 1^2 - 2|-1| = -1$ and $f(1) = 1^2 - 2|1| = -1$ are the relative minima.
That is, -1 is the relative minimum, which occurs at $x = \pm 1$.

4. For each given function,
(i) use the Second Derivative Test, when applicable, to find the relative extrema;
(ii) find intercepts and points of inflection, when possible;
(iii) find the intervals on which it is increasing and the interval on which it is decreasing;
(iv) find the intervals on which it is concave up and the intervals on which it is concave down;
and **(v)** sketch the graph.

(a) $f(x) = x^3 + 3x^2 + 3x + 1$
(b) $f(x) = 6x^5 - 10x^3$
(c) $f(x) = \frac{x}{x^2+2}$
(d) $f(x) = 2x - x\ln(x)$

Answer:

Note that there are 2 steps
(1) Find critical numbers c_j
(2) Check sign of $f''(c_j)$:
- If $f''(c_j) < 0$, $f(c_j)$ is a relative maximum.
- If $f''(c_j) > 0$, $f(c_j)$ is a relative minimum.
(If $f''(c_j) = 0$, no conclusion – use the First Derivative Test.)

(a) $f(x) = x^3 + 3x^2 + 3x + 1$

$$f'(x) = 3x^2 + 6x + 3, \qquad f''(x) = 6x + 6.$$

(i) Critical numbers:

$f'(x) = 3x^2 + 6x + 3 = 0 \quad \Rightarrow \quad 3(x+1)^2 = 0 \quad \Rightarrow \quad x = -1$ is the only critical number.

From the Second Derivative Test,

$$f''(-1) = 6(-1) + 6 = 0$$

so we cannot conclude anything. The First derivative will be used instead,

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f'(x) = 3(x+1)^2$	$+$	$+$

Therefore, from the table above, the *First Derivative Test* implies that there is **no relative extremum** since the sign of $f'(x)$ is always positive.

(ii) Find intercepts and points of inflection:
- The y-intercept(x=0) is at $y = f(0) = 0^3 + 30^2 + 3(0) + 1 = 1 \Rightarrow\ y = 1$.
- The x-intercept(y=0) is at $0 = f(x) = (x+1)^3 \Rightarrow\ x = -1$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 6x + 6 = 0 \qquad \Longrightarrow \qquad x = -1.$$

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f''(x) = 6(x+1)$	$-$	$+$

Since the sign of $f''(x)$ changes at the $x = -1$, it gives an inflection point. When $x = -1$, $y = f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0$. Hence, the inflection point is $(-1, 0)$.
(iii) Since $f'(x) = 3(x+1)^2 > 0$for all $x \neq -1$, and $f'(x) = 0$ at $x = -1$, $f(x)$ is increasing on $(-\infty, \infty)$.

(iv) Concavity:

$$f''(x) = 6x + 6 > 0 \;\Rightarrow\; x > -1 \text{ and } \quad f''(x) = 6x + 6 < 0 \;\Rightarrow\; x < -1.$$

That is, $f(x)$ is concave up on $(-1, \infty)$ and $f(x)$ is concave down on $(-\infty, -1)$.

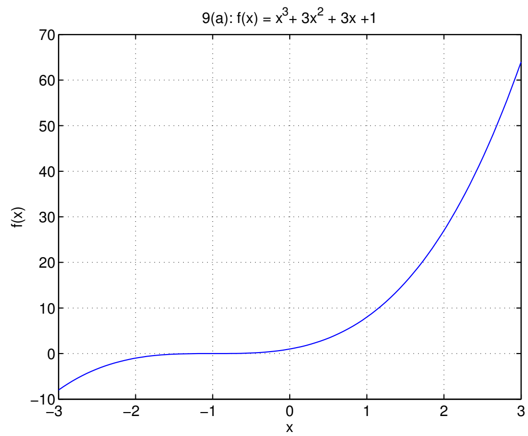

Figure 2: (a): $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = 6x^5 - 10x^3$

(i) Critical numbers:

$$f'(x) = 30x^4 - 30x^2 = 30x^2(x^2-1) = 30x^2(x-1)(x+1) = 0 \;\Rightarrow$$

$x = 0, -1, 1$ are the only critical numbers. From

$$f''(x) = 120x^3 - 60x = 60x(2x^2-1)$$

and from the Second Derivative Test,
$f''(0) = 0 \Longrightarrow$, no conclusion
$f''(-1) = -60 < 0 \Longrightarrow$, $f(-1) = -6 + 10 = 4$ is a relative maximum
$f''(1) = 60 > 0 \Longrightarrow$, $f(1) = 6 - 10 = -4$ is a relative minimum.
The First Derivative Test will be used to identify the critical number $x = 0$ as shown in the table. Since the sign of $f'(x)$ does not change at $x = 0$, $x = 0$ does not give an extremum.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
Sign of $f'(x) = 30x^2(x-1)(x+1)$	$+$	$-$	$-$	$+$
$x^2(x-1)(x+1)$	$(+)(-)(-)$	$(+)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

(ii) Find intercepts and points of inflection:
- The y-intercept(x=0) is at $y = f(0) = 6(0)^5 - 10(0)^3 = 0 \Rightarrow \; y = 0$.
- The x-intercept(y=0) occurs when $0 = f(x) = x^3(6x^2 - 10)$
$\Rightarrow$ at $x = 0$, $x = -\sqrt{5/3}$, $x = \sqrt{5/3}$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 60x(2x^2-1) = 0 \qquad \Longrightarrow \quad x = 0, -\sqrt{1/2}, \sqrt{1/2}.$$

	$x \in (-\infty, -\sqrt{\frac{1}{2}})$	$x \in (-\sqrt{\frac{1}{2}}, 0)$	$x \in (0, \sqrt{\frac{1}{2}})$	$x \in (\sqrt{\frac{1}{2}}, \infty)$
Sign of $f''(x) = 60x(2x^2-1)$	$-$	$+$	$-$	$+$
$(x)(x-\sqrt{1/2})(x+\sqrt{1/2})$	$(-)(-)(+)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{1/2}, \sqrt{1/2}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{1/2}) = 7\sqrt{2}/4$, $f(\sqrt{1/2}) = -7\sqrt{2}/4$,
the inflection points are $(0,0)$, $(-\sqrt{1/2}, 7\sqrt{2}/4)$,$(\sqrt{1/2}, -7\sqrt{2}/4)$.

(iii) From the table in (i), we see that
$f(x)$ is increasing ($f'(x) > 0$) on $(-\infty, -1) \cup (1, \infty)$ and
$f(x)$ is decreasing ($f'(x) < 0$) on $(-1, 1)$.

(iv) From the table in (ii), we see that
$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$ and
$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$.

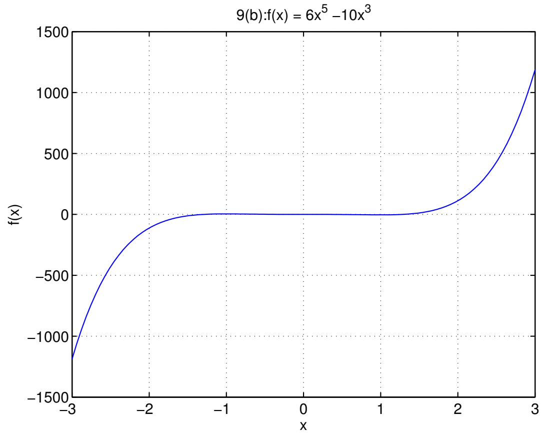

Figure 3: (b): $f(x) = 6x^5 - 10x^3$

(c) $f(x) = \frac{x}{x^2+2}$

(i) Critical numbers:

$$f'(x) = \frac{(x^2+2) - 2x^2}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2} = 0 \ \Rightarrow \quad 2 - x^2 = 0$$

$x = -\sqrt{2}, \sqrt{2}$ are the only critical numbers. From

$$f''(x) = \frac{2x(x^2+2)(x^2-6)}{(x^2+2)^4} = \frac{2x(x^2-6)}{(x^2+2)^3}$$

and from the Second Derivative Test,

$f''(-\sqrt{2}) = \frac{\sqrt{2}}{8} > 0 \Longrightarrow, f(-\sqrt{2}) = \frac{-\sqrt{2}}{(-\sqrt{2})^2+2} = -\sqrt{2}/4$ is a relative minimum

$f''(\sqrt{2}) = -\frac{\sqrt{2}}{8} < 0 \Longrightarrow, f(\sqrt{2}) = \frac{\sqrt{2}}{(\sqrt{2})^2+2} = \sqrt{2}/4$ is a relative maximum.

(ii) Find intercepts and points of inflection:

- The y-intercept(x=0) is at $y = f(0) = 0 \Rightarrow \ y = 0$.
- The x-intercept(y=0) occurs when $0 = f(x) \Rightarrow$ at $x = 0$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3} = 0 \qquad \Longrightarrow \quad x = 0, -\sqrt{6}, \sqrt{6}.$$

Since the denominator $(x^2+2)^3 > 0$, the sign of $f''(x)$ depends on the numerator $2x(x^2-6) = 2x(x-\sqrt{6})(x+\sqrt{6})$ as shown in the table below.

	$x \in (-\infty, -\sqrt{6})$	$x \in (-\sqrt{6}, 0)$	$x \in (0, \sqrt{6})$	$x \in (\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3}$ $(2x)(x-\sqrt{6})(x+\sqrt{6})$	$-$ $(-)(-)(+)$	$+$ $(-)(-)(+)$	$-$ $(+)(-)(+)$	$+$ $(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{6}, \sqrt{6}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{6}) = \frac{-\sqrt{6}}{(-\sqrt{6})^2+2} = -\sqrt{6}/8$, $f(\sqrt{1/2}) = \frac{\sqrt{6}}{(\sqrt{6})^2+2} = \sqrt{6}/8$,
the inflection points are $(0,0)$, $(-\sqrt{6}, -\sqrt{6}/8)$,$(\sqrt{6}, \sqrt{6}/8)$.

(iii) From

$$f'(x) = \frac{2-x^2}{(x^2+2)^2} = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2},$$

the denominator is always positive, so the sign of $f'(x)$ depends on the numerator:
$(\sqrt{2}-x)(\sqrt{2}+x)$.

	$x \in (-\infty, -\sqrt{2})$	$x \in (-\sqrt{2}, \sqrt{2})$	$x \in (\sqrt{2}, \infty)$
Sign of $f'(x) = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2}$	$-$	$+$	$-$
$(\sqrt{2}-x)(\sqrt{2}+x)$	$(+)(-)$	$(+)(+)$	$(-)(+)$

Hence, $f(x)$ is increasing ($f'(x) > 0$) on $(-\sqrt{2}, \sqrt{2})$ and
$f(x)$ is decreasing ($f'(x) < 0$) on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

(iv) From the table in (ii), we see that
$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$ and
$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$.

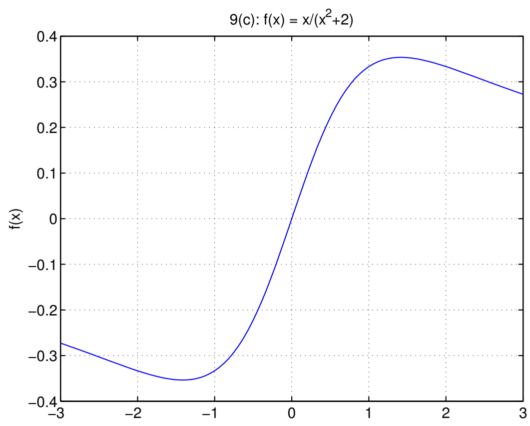

Figure 4: (c): $f(x) = \frac{x}{x^2+2}$

(d) $f(x) = 2x - x\ln(x)$

First note that the domain of f is
$$\mathcal{D} = (0, \infty),$$
since $\ln(x)$ is only well-defined for $x > 0$.

(i) Critical numbers: $f'(x) = 2 - \left[x\frac{1}{x} + \ln(x)\right] = 1 - \ln(x)$.
$$f'(x) = 1 - \ln(x) = 0 \ \Rightarrow \quad \ln(x) = 1 \ \Rightarrow \quad x = e^1.$$
$x = e$ is the only critical number. From
$$f''(x) = -\frac{1}{x}$$
and from the Second Derivative Test,
$f''(e) = -\frac{1}{e} < 0 \Longrightarrow$, $f(e) = 2e - e\ln(e) = 2e - e = e$ is a relative maximum.

(ii) Find intercepts and points of inflection:

- There is no y-intercept, since $x = 0$ is not in the domain of the function $f(x) = 2x - x\ln(x)$ ($\ln(x)$ is not well-defined at $x = 0$).
- The x-intercept(y=0) occurs when $0 = f(x) \Rightarrow 2x - x\ln(x) = x(2 - \ln(x)) = 0 \Rightarrow x = 0$ or $\ln(x) = 2 \Rightarrow$ at $x = 0, x = e^2$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$: notice that
$$f''(x) = -\frac{1}{x} \neq 0 \qquad \Longrightarrow \qquad \text{There is no inflection point.}$$

(iii) From $f'(x) = 1 - \ln(x)$,

- $f'(x) > 0$ when $1 - \ln(x) > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \Rightarrow f(x)$ is **increasing** on $(-\infty, e]$.
- $f'(x) < 0$ when $1 - \ln(x) < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \Rightarrow f(x)$ is **decreasing** on $[e, \infty)$.

(iv) From $f''(x) = -\frac{1}{x}$,
$f(x)$ is concave down ($f''(x) < 0 \Leftrightarrow x > 0$) on $(0, \infty)$. Note that $f(x)$ is not concave up, since $f''(x) > 0 \Leftrightarrow x < 0$, but $x < 0$ is not in the domain of $f(x)$: $\mathcal{D} = (0, \infty)$.

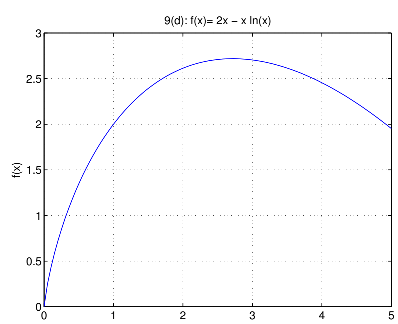

Figure 5: (d): $f(x) = 2x - x\ln(x)$

5. Let $X, Y_1, Y_2 \subseteq \mathbb{R}$. Consider the relations f_1, f_2 defined by

$$f_1 = \left\{ (x,y) \in X \times Y_1 \middle| \quad y = \sqrt{\frac{x+1}{2x-3}} \right\}, \quad f_2 = \left\{ (x,y) \in X \times Y_2 \middle| (x,-y) \in f_1 \right\}.$$

Sketch the curve of the relation $f = f_1 \cup f_2$.

Answer:
(Details to be added...symmetry/critical numbers/increasing& decreasing/ concavity...)

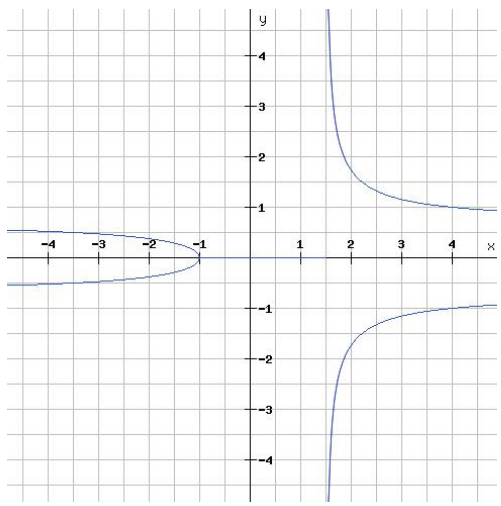

Figure 6: $f = f_1 \cup f_2 = \{(x,y) | y^2 = \frac{x+1}{2x-3}\}$

6. Find the vertical and horizontal asymptotes (if any) of $f(x) = \frac{x^2-1}{2x}$ and show that $y = \frac{x}{2}$ is a slant asymptote of f.

Answer:(Details to be added...)

- Vertical asymptote: $x = 0$
- Horizontal asymptote: None
- Slant asymptote $y = \frac{x}{2}$. We have to show that $\lim\limits_{x\to\infty} [f(x) - \frac{x}{2}] = 0$.

$$\lim_{x\to\infty} [f(x) - \frac{x}{2}] = \lim_{x\to\infty} \left(\frac{x^2-1}{2x} - \frac{x}{2} \right) = \lim_{x\to\infty} \left(\frac{x^2-1-x^2}{2x} \right) = \lim_{x\to\infty} \frac{-1}{2x} = 0$$