Solution: Quiz 6

1. Let $A = \{1, 2\}$ and $B = \{2\}$. Let $\mathcal{P}(A)$ and $\mathcal{P}(B)$ be the power sets of A and B, respectively. Define

$$X = \{x \in \mathcal{P}(A) | \ x \cup \emptyset \neq \emptyset\}, \qquad \text{and} \qquad Y = \{y \in \mathcal{P}(A) | \ y \cap \{2\} = \emptyset\},$$

and define functions $F : X \to Y$ and $G : \mathcal{P}(B) \to \{1, 2\}$, respectively, by

$$F = \{(x, y) \in X \times Y | \ [\text{number of elements in } x] - [\text{number of elements in } y] = 1\},$$

$$G(w) = \begin{cases} 1, & \text{for } w \cup \emptyset = \emptyset \\ 2, & \text{for } w \cup \emptyset \neq \emptyset \end{cases}, \qquad w \in \mathcal{P}(B)$$

Draw arrow diagrams for F, G, $G \circ F$ and determine the domain of $G \circ F$.

Solution: First note that

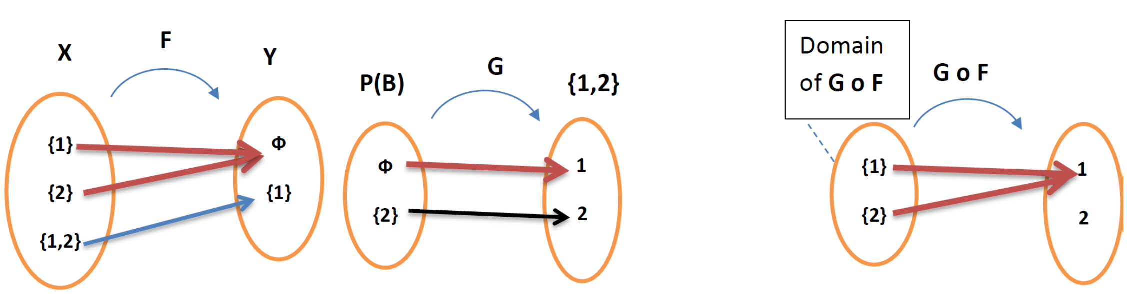
$$\begin{aligned} \mathcal{P}(B) &= \{\emptyset, \{2\}\} \\ \mathcal{P}(A) &= \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \\ X &= \{x \in \mathcal{P}(A) | \ x \cup \emptyset \neq \emptyset\} = \{\{1\}, \{2\}, \{1, 2\}\} \quad \Leftarrow \quad \text{every element in } \mathcal{P}(A) \text{ except } \emptyset \\ Y &= \{y \in \mathcal{P}(A) | \ y \cap \{2\} = \emptyset\} = \{\emptyset, \{1\}\} \quad \Leftarrow \quad \text{every element in } \mathcal{P}(A) \text{ that does not contain } 2. \end{aligned}$$

▶ To list the elements in F, consider $x \in X = \{\{1\}, \{2\}, \{1, 2\}\}$ and $Y = \{\emptyset, \{1\}\}$.
- For $x = \{1\}$, since number of elements in x is 1, we must have number of elements in y equal to 0, which implies $y = \emptyset$. I.e. $\boxed{(x, y) = (\{1\}, \emptyset) \in F \text{ or } F(\{1\}) = \emptyset}$.
- For $x = \{2\}$, since number of elements in x is 1, we must have number of elements in y equal to 0, which implies $y = \emptyset$. I.e. $\boxed{(x, y) = (\{2\}, \emptyset) \in F \text{ or } F(\{2\}) = \emptyset}$.
- For $x = \{1, 2\}$, since number of elements in x is 2, we must have number of elements in y equal to 1, which implies $y = \{1\}$. I.e. $\boxed{(x, y) = (\{1, 2\}, \{1\}) \in F \text{ or } F(\{1, 2\}) = \{1\}}$.

Hence, the domain $D_F = \{\{1\}, \{2\}, \{1, 2\}\}$ and the range $R_F = \{\emptyset, \{1\}\}$. The arrow diagram of F is given below.

▶ From the definition of $G(w)$, consider $w \in D_G = \mathcal{P}(B) = \{\emptyset, \{2\}\}$.
- For $w = \emptyset$, $w \cup \emptyset = \emptyset \quad \Rightarrow \quad G(\emptyset) = 1$, and
- For $w = \{2\}$, $w \cup \emptyset = \{2\} \neq \emptyset \quad \Rightarrow \quad G(\{2\}) = 2$,

The arrow diagram of G is given below.



▶ Since $R_F \cap D_G = \{\emptyset, \{1\}\} \cap \{\emptyset, \{2\}\} = \{\emptyset\} \neq \emptyset$, then we can define $G \circ F$

$$(G \circ F)(x) = G(F(x))$$

where x satisfies the conditions (i) $x \in D_F$ and (ii) $F(x) \in D_G$:
(i) $x \in D_F = \{\{1\}, \{2\}, \{1,2\}\}$ and
(ii) $y = F(x) \in D_G = \mathcal{P}(B) = \{\emptyset, \{2\}\} \quad \Rightarrow \quad y = \emptyset \quad \Rightarrow \quad x = \{1\}, \{2\}$.
That is, x that can be used for $(G \circ F)(x)$ must satisfy

$$x \in \{\{1\}, \{2\}, \{1,2\}\} \ \cap \ \{\{1\}, \{2\}\} = \{\{1\}, \{2\}\} \qquad \Rightarrow \qquad D_{G \circ F} = \{\{1\}, \{2\}\}$$

and

$$(G \circ F)(\{1\}) = G(F(\{1\})) = G(\emptyset) = 1 \qquad \text{and} (G \circ F)(\{2\}) = G(F(\{2\})) = G(\emptyset) = 1,$$

which is shown in the diagram above and its domain is $\boxed{D_{G \circ F} = \{\{1\}, \{2\}\}}$. ■