Example 3.6. Find **all possible values** of k so that $x \in (-\infty, -3/2] \cup [4, \infty)$ satisfies the following inequality:

$$-2x^2 + \frac{5}{2}xk + 3k^2 \leq 0.$$

Solution: In this problem, it is possible to find many different values of k. Note that, in order to have $x \in (-\infty, -3/2] \cup [4, \infty)$ satisfying the given inequality, we may have a larger solution set that contains $(-\infty, -3/2] \cup [4, \infty)$.
The formula for factoring the quadratic polynomial can be used to get

$$-2x^2 + \frac{5}{2}xk + 3k^2 = -2(x + 3/4k)(x - 2k).$$

So, solving $-2x^2 + \frac{5}{2}xk + 3k^2 \leq 0$ is equivalent to solving $-2(x + 3/4k)(x - 2k) \leq 0$ or

$$2(x + 3/4k)(x - 2k) \geq 0$$

which can be done by considering
$2(x + \frac{3}{4}k)(x - 2k) = 0$ or $x = -\frac{3}{4}k$ or $x = 2k$ for subdividing the intervals.

- Case I: for $k > 0$, we have $-\frac{3}{4}k < 2k$ the solution $x \in (-\infty, -\frac{3}{4}k] \cup [2k, \infty)$, which will contain $(-\infty, -\frac{3}{2}] \cup [4, \infty)$ when

$$-\frac{3}{4}k \geq -\frac{3}{2} \qquad \text{and} \qquad 2k \leq -\frac{3}{2} \qquad \Rightarrow \qquad k \leq 2.$$

That is, $k \in (0, 2]$ in this case.

- Case II: for $k = 0$, the inequality becomes $-2x^2 \leq 0$ or $x^2 \geq 0$, which is always true for any real number x. I.e., the solution set is $\mathbb{R}$, which contains $(-\infty, -\frac{3}{2}] \cup [4, \infty)$.
That is, $k = 0$ is a possible value for this problem.

- Case III: for $k < 0$, we have $-\frac{3}{4}k > 2k$ the solution $x \in (-\infty, 2k] \cup [-\frac{3}{4}k, \infty)$, which will contain $(-\infty, -\frac{3}{2}] \cup [4, \infty)$ when

$$2k \geq -\frac{3}{2} \qquad \text{and} \qquad -\frac{3}{4}k \leq 4 \qquad \Rightarrow \qquad k \geq -\frac{3}{4}$$

That is, since $k < 0$, then $k \in [-\frac{3}{4}, 0)$ in this case.
From the 3 cases I, II, and III, the possible values of k are in the interval $[-\frac{3}{4}, 0) \cup \{0\} \cup (0, 2] = [-\frac{3}{4}, 2]$.
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